Physics 402
Fall 2022
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Discussion Worksheet for 5 October, 2022

1. (a) Write down the Hamiltonian for two noninteracting identical particles in the infinite square well of width a. Ignore spin.
(b) Verify that the Fermion ground state given below is an eigenfunction of the Hamiltonian and find the eigenvalue.

$$\boldsymbol{\Psi(x_1, x_2) =}$$
$$\begin{cases} \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right] & \text{for } 0 < x_1 < a \text{ and } 0 < x_2 < a \\ 0 & \text{otherwise} \end{cases}$$

a) $$\hat{\mathcal{H}}^0 = \frac{-\hbar^2}{2m}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + V(x_1) + V(x_2) \quad \text{with} \quad V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & x < 0,\ x > a \end{cases}$$

This is the Hamiltonian of two non-interacting particles in the same infinite square well.

b) Put $\Psi(x_1, x_2)$ into the TISE $\hat{\mathcal{H}}^0 \Psi = E\Psi$. The second derivatives basically return the same wave function, with a factor out in front

$$\frac{d^2}{dx_1^2}\Psi = \frac{\sqrt{2}}{a}\left[-\left(\frac{\pi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - (-1)\left(\frac{2\pi}{a}\right)^2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

Group in pairs with common factor

$$\frac{d^2}{dx_2^2}\Psi = \frac{\sqrt{2}}{a}\left[-\left(\frac{2\pi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - (-1)\left(\frac{\pi}{a}\right)^2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

Add them into the TISE:

$$\hat{\mathcal{H}}^0 \Psi = \frac{-\hbar^2}{2m}\left\{-\left(\frac{\pi}{a}\right)^2\Psi - \left(\frac{2\pi}{a}\right)^2\Psi + 0 + 0\right\} \quad \text{for } 0<x<a$$

$$= +\frac{\hbar^2\pi^2}{2ma^2}\left\{1 + 4\right\}\Psi$$

$$= \frac{5\pi^2\hbar^2}{2ma^2}\Psi \qquad \text{Hence } E = \frac{5\pi^2\hbar^2}{2ma^2}$$

Yes, Ψ is an eigenfunction of the two-particle Hamiltonian with eigenvalue $E = \frac{5\pi^2\hbar^2}{2ma^2}$

2. Find the next excited state eigenfunction and eigenvalue for two identical Fermions in the infinite square well. Again ignore spin.

The given WF corresponds to the "12" state with $E_{12} = 5K$ with $K \equiv \frac{\pi^2 \hbar^2}{2ma^2}$

The states "11" and "22", etc. are not allowed because they put both Fermions in the same exact state (i.e. the same exact list of quantum numbers) $\Rightarrow \Psi = 0$

The next few possibilities are "13", "23", ~~"24"~~ "14":

$$E_{13} = (3^2+1^2)K = 10K$$

$$E_{23} = (2^2+3^2)K = 13K$$

$$E_{14} = (1^2+4^2)K = 17K$$

So the next excited state will be "13"

$$\Psi_{13}(x_1, x_2) = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) - \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

Note that $P\Psi(x_1, x_2) = \Psi(x_2, x_1) = -\Psi(x_1, x_2)$,

as expected for two identical Fermions with overlapping wave-functions.